ON THE STABILITY OF STOCHASTIC SYSTEMS IN THE LARGE

(OB USTOICHIVOSTI V TSELOM STOKHASTICHESKIKH SISTEM)

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The problem of stability in probability of stochastic systems of differential equations in the large is considered. A stability criterion based on the use of two Liapunov functions [1] is given.

The idea of using two Liapunov functions is due to Chetaev [2]. In the case of ordinary differential equations, the stability criteria constructed by using two functions were proved in [3].

The theorem proved below for stochastic systems is analogous to that which was proved for ordinary differential equations [4].

1. Let the differential equations of perturbed motion be

$$dx / dt = f(t, x, y(t)) \tag{1.1}$$

where x is an n-dimensional vector of the phase coordinates of the system, the vector function $f = \{f_1, \ldots, f_n\}$ is continuous in all the variables in the domain

$$-\infty < x_i < +\infty, \qquad t \geqslant 0, \qquad y \in Y \tag{1.2}$$

and satisfies the Lipschitz conditions in the x_j, y variables in this domain and is bounded for all $y \in Y$ in each finite domain

$$\|x\| \leqslant N \ (\|x\| = \max\{|x_1|, \ldots, |x_n|\}).$$

The function $y(t)$ describes a Markov random process [5] which we shall assume to be either purely discontinuous ([6], p.292) or continuous ([6], p.284). Let us limit ourselves to the consideration of only the scalar function $y(t)$. The results are generalized to the case when $y(t)$ is an m-dimensional vector without essential changes in the reasoning.

Under certain sufficiently broad assumptions [5], a continuous Markov stochastic process can be considered as the solution of the diffusion equation

$$dy / dt = m(t, y) + \sigma(t, y)\, dq / dt$$

where $q(t)$ is a Wiener process, i.e. a Gaussian process with independent increments satisfying conditions

$$M[q(t_2) - q(t_1)] = 0, \qquad M[q(t_2) - q(t_1)]^2 = |t_1 - t_2|$$

(for any $t_1 \geqslant 0,\ t_2 \geqslant 0$).

Questions of the stability of stochastic systems have been considered in a number of works [7 to 11]. The definition and notation used in [11] are used herein. The case when $y(t)$ is a homogeneous Markov chain with a finite number of states is considered in [11], however, the definitions and results which are used later remain valid even under more general assumptions.

Let us present certain definitions.

Definition 1.1. The solution $x = 0$ of the system (1.1) will be called probabilistically stable if for any number $\varepsilon > 0$, $p > 0$ as small as desired, it is possible to indicate a $\delta > 0$ such that for arbitrary initial data satisfying condition

$$\| x(t_0) \| \leqslant \delta, \qquad y(t_0) \in Y \tag{1.3}$$

the inequality

$$P\{\| x(t) \| < \varepsilon, \quad t \geqslant t_0 \,/\, \| x(t_0) \| \leqslant \delta, \quad y(t_0) \in Y\} > 1 - p \tag{1.4}$$

will be valid.

The symbol $P\{A/B\}$ denotes the conditional probability of the event A.

Equations (1.1) generate a probabilistic Markov process $\{x(t),\ y(t)\}$ which we may consider separable ([5], p.53). Then the expression on the left-hand side of the inequality (1.4) has meaning.

Definition 1.2. The solution $x = 0$ of the system (1.1) will be called asymptotically stable in the large, if it is stable in the sense of definition (1.1) and if, no matter what the bounded domain $\| x \| \leqslant H_0$ and the numbers $\gamma > 0,\ 0 < p < 1,\ 0 < q < 1$, it is possible to indicate a bounded domain $\|x\| < H_1$ and a number $T > 0$ such that conditions

$$P\{\| x(t) \| < H_1 \quad t > t_0 \,/\, \| x_0 \| \leqslant H_0,\ y_0 \in Y\} > 1 - p \tag{1.5}$$

$$P\{\| x(t) \| > \gamma, \quad t > t_0 + T \,/\, \| x_0 \| \leqslant H_0, y_0 \in Y\} > 1 - q \tag{1.6}$$

will be satisfied.

The meaning of this definition is the following: if the solution $x = 0$ is asymptotically stable in the large, then for any initial condition (x_0, y_0) the motion $x(t)$ will be in a certain bounded domain $\|x\| < H_1$ at $t > t_0$ with a probability as close to 1 as desired. Hence, starting with a certain sufficiently large time $t_0 + T$, the motion trajectory will drop into as small a neighborhood of the origin of coordinates as desired and will remain there for all $t > t_0 + T$ with probability as close to 1 as desired.

Note 1.1. Definitions (1.1) and (1.2) agree with the corresponding definitions of [11] under the condition of discontinuity of realization assumed there.

Note 1.2. The stability condition analogous to inequality (1.4) can be selected from Definition 1.1 in the form

$$\lim\{P[\sup \| x(t) \|, \quad t_0 \leqslant t < \infty] > \varepsilon \,/\, x(t_0) = x_0\} = 0 \quad \textbf{for} \quad x_0 \to 0 \tag{1.7}$$

Such a definition for stability is given by Khas'minskii [10].

2. The sufficient conditions for stability of stochastic systems may be given in a form analogous to the theorems of the second Liapunov method for ordinary differential equations [10 and 11]. In particular, if a positive-definite function $v(t,x,y)$ admitting an infinitely small upper and an infinitely large lower limit ([12], p.36) exists for Equations (1.1) and whose derivative $dM[v]/dt$ is negative definite * by virtue of the system (1.1), then the solution $x = 0$ is asymptotically stable in the large.

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* See opposite page.

However, just as for determined systems, it is sometimes possible to construct a function for stochastic systems which will be positive definite, and will admit an infinitely small upper and an infinitely large lower limit, but whose derivative is only a negative function. Therefore, it is impossble to use the theorem mentioned above in this case.

Theorem 4 from [4] can be applied to determined systems in a similar situation. The theorem presented below indicates the condition when the asymptotic stability in the large of a stochastic system is guaranteed by a function with a negative derivative. This theorem is based on the use of two functions. The idea of using two Liapunov functions is due to Chetaev [2]. Theorems on asymptotic stability for ordinary differential equations are proved in [3] by using two functions.

Let us introduce some definitions.

Definition 2.1. Let G be an open domain in the space $\{x_1\}$. Let the function $\psi(t,x,y)$ be designated positive-definite in the domain $G \times Y$ if for any numbers $\varepsilon > 0$, $H > \varepsilon > 0$, a $\delta > 0$ can be indicated such that

$$\psi(t, x, y) \geqslant \delta \quad \textbf{for} \quad t \geqslant 0, \ \{x, y\} \in G \times Y; \quad \varepsilon \leqslant \|x\| \leqslant H \tag{2.1}$$

Definition 2.2. Let us say that the function $F(t,x,y)$ satisfies conditions $A(G)$ if

a) It is bounded for all $t \geqslant 0$ in any finite domain

$$\|x\| \leqslant H, \qquad y \in Y \tag{2.2}$$

b) The derivative $dM[F]/dt$ of the function $F(t,x,y)$ by virtue of the system is bounded in any domain (2.2), i.e. $|dM[F]/dt| \leqslant K$.

c) By virtue of Equations (1.1) the derivative $dM[F]/dt$ is positive definite in the domain $G \times Y$.

The following theorem is valid.

Theorem 2.1. Let these conditions be satisfied for Equations (1.1) defined in the domain (1.2).

a) There exists a positive-definite function $v(t,x,y)$ admitting an infinitely small upper and an infinitely large lower limit, i.e. conditions

$$\lim v(t, x, y) = 0 \quad \textbf{for} \quad x \to 0, \qquad \lim v(t, x, y) = \infty \quad \textbf{for} \quad x \to \infty$$

are valid uniformly in t and y.

b) The derivative $dM[v]/dt$ satisfies condition

$$dM[v] / dt \leqslant -\Phi(x) \leqslant 0$$

by virtue of the system (1.1), where the function $\Phi(x)$ is continuous in the domain $\|x\| \leqslant H$.

c) The set Q of points where $\Phi(x) = 0$ (except the point $x = 0$) is internal with respect to a certain open domain $G \supset Q$.

d) There exists a function $F(t,x,y)$ satisfying conditions $A(G)$.

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* The derivative $dM[v]/dt$ has the following meaning at the point $t = \tau$, $x = \xi$, $y = \eta$ ([11], English version p.1229)

$$\frac{dM[v]}{dt} = \lim_{t \to \tau+0} \frac{1}{t-\tau} \{M[v(t, x(t), y(t)) - v(\tau, \xi, \eta)] / \xi, \eta\}$$

and is determined by the infinitesimal operator [13] of the process $\{x(t), y(t)\}$.

The quantity $dM[v]/dt$ may be interpreted as the average value of the derivative of the function $v(t,x,y)$ along all realizations $\{x(\omega,t), y(\omega,t)\}$, issuing from the point $\{\xi,\eta\}$ at the time τ.

Then the unperturbed motion $x = 0$ of the system (1.1) is asymptotically stable in probability in the large.

P r o o f . Let us take the arbitrary numbers $H_0 > 0$, $\epsilon < H_0$, $0 < \mu < 1$, $0 < \gamma < 1$. By means of these numbers let us determine two numbers $H > 0$ and $\epsilon_1 > 0$ from conditions

$$[\sup v(t_0, x_0, y_0), \ \|x_0\| \leqslant H_0, y_0 \in Y] < \mu \, [\inf v(t, x, y), \|x\| \geqslant H, \ y \in Y, \ \forall t \geqslant t_0] \qquad (2.3)$$

$$[\sup v(t, x, y), \ \|x\| \leqslant \varepsilon_1, y \in Y, t \geqslant t_0] < \\ < {}^1/_2 \gamma \, [\inf v(t, x, y), \ \|x\| \geqslant \varepsilon, \ y \in Y, \ t \geqslant t_0] \qquad (2.4)$$

As has been shown in [11], by virtue of the conditions of the theorem, the validity of the relations

$$P\{\|x(t)\| < H, \ t \geqslant t_0 \mid \|x_0\| \leqslant H_0, \ y_0 \in Y\} > 1 - \mu \qquad (2.5)$$

$$P\{\|x(t)\| > \varepsilon, \ t \geqslant \tau, \mid \|x(\tau)\| \leqslant \varepsilon_1, y(\tau) \in Y\} > 1 - {}^1/_2 \gamma \qquad (2.6)$$

follows from these inequalities.

Now, it is sufficient for us to show that for any point $\{x_0, y_0\}$ from the domain

$$\|x_0\| \leqslant H_0, \qquad y_0 \in Y$$

a time $\tau > t_0$, is found for which the inequality

$$P\{\|x(\tau)\| \leqslant \varepsilon_1 \mid \|x_0\| \leqslant H_0, y_0 \in Y\} > 1 - {}^1/_2 \gamma - \mu \qquad (2.7)$$

is satisfied since in this case condition

$$P\{\|x(t)\| < \varepsilon, \ t \geqslant \tau \mid \|x_0\| \leqslant H_0, y_0 \in Y\} > 1 - \mu - \gamma \qquad (2.8)$$

would be satisfied because of the stability in probability and (2.6).

Let us prove the inequality (2.7) by contradiction. Let (2.7) not be satisfied, then a point $\{x_0, y_0\}$ is found such that for each $t > t_0$ the inequality

$$P\{\|x(t)\| > \varepsilon_1, \ \|x_0\| \leqslant H_0, y_0 \in Y\} > {}^1/_2 \gamma + \mu \qquad (2.9)$$

will be valid.

By virtue of the stability of the system, a $\delta > 0$ can now be determined by means of the numbers $\gamma > 0$ and $\epsilon_1 > 0$, such that for each $t > t_0$ condition

$$P\{\|x(t)\| > \varepsilon_1 \mid \|x(t_0)\| \leqslant \delta, \ y(t_0) \in Y\} < {}^1/_8 \gamma \qquad (2.10)$$

would be satisfied.

The relation

$$P(\{\|x(t)\| > \delta, \ t \geqslant t_0 \mid x_0, y_0\} > {}^1/_4 \gamma + \mu \qquad (2.11)$$

results from (2.9) and (2.10).

In fact, if (2.11) were not satisfied, then condition

$$P\{\|x(t)\| > \delta, \ t \geqslant t_0 \mid x_0, y_0\} = \lim_{\theta \to \infty} P\{\|x(t)\| > \delta, \quad t_0 \leqslant t \leqslant \theta \mid x_0, y_0\} < {}^1/_4 \gamma + \mu$$

would hold.

But it is then possible to indicate so large a value $\theta_0 > t_0$ that

$$P\{\|x(t)\| > \delta, \ t_0 \leqslant t \leqslant \theta_0 \mid x_0, y_0\} < {}^3/_8 \gamma + \mu \qquad (2.12)$$

Let τ denote the first time the trajectory $x(t)$ hits the surface $\|x\| = \delta$, we will then have for $t > \theta_0$

$$P\{\|x(t)\| > \varepsilon_1 \mid x_0, y_0\} \leqslant \qquad (2.13)$$

$$\leqslant \int\limits_{\eta \in Y, \ \|\xi\| = \delta} P\{\|x(t)\| > \varepsilon_1 \mid x(\tau_\delta) = \xi, \ y(\tau_\delta) = \eta\} \, P\{x(\tau_\delta) \in d\xi,$$

$$y(\tau_\delta) \in d\eta,\ \tau_\delta \leqslant \theta_0 / x_0, y_0\} + P\{\|x(s)\| > \delta,\ t_0 \leqslant s \leqslant \theta_0 / x_0; y_0\} < \quad (2.13)\ \text{cont.}$$
$$< {}^1/_8\gamma + {}^3/_8\gamma + \mu = {}^1/_2\gamma + \mu$$

Here the probabilistic stability property is used which is analogous, in a certain sense, to the strict Markov property [13], i.e. the inequality (1.4) is used under the assumption that t_0 is a random time independent of the future.

The inequality obtained contradicts condition (2.9). Hence, the validity of the relation (2.11) follows. Comparing (2.5) and (2.11), we obtain the following estimate:

$$P\{\delta < \|x(t)\| < H,\ t \geqslant t_0 / x_0, y_0\} > {}^1/_4\gamma \tag{2.14}$$

Let us show that (2.14) contradicts the conditions of the theorem. To do this, let us consider the domains D and E defined by the equalities

$$D = G \cap \{\delta < \|x\| < H\}, \qquad E = \{\delta < \|x\| < H\} \setminus \dot{D}$$

Then positive constants k^2 and m^2 can be indicated such that the relations

$$\left[\sup \frac{dM[v]}{dt},\ x \in E,\ y \in Y,\ t \geqslant t_0\right] = -k^2 \tag{2.15}$$

$$\left[\inf \frac{dM[F]}{dt},\ x \in D,\ y \in Y,\ t \geqslant t_0\right] = m^2 \tag{2.16}$$

will be valid.

Together with the process $\{x(t),y(t)\}$, which is a solution of Equations (1.1), let us consider the auxiliary stochastic process $\{x^*(t),y^*(t)\}$.

Let us assume that the realizations $\{x^*(t,\omega),y^*(t,\omega)\}$ of the stochastic process $\{x^*(t),y^*(t)\}$ exist and agree with the corresponding realizations $\{x(t,\omega),y(t,\omega)\}$ of the process $\{x(t),y(t)\}$ only until $\delta < \|x(t,\omega)\| < H$. If $t = t(\omega)$ is the time of the first emergence of the realization beyond the boundary of the domain under consideration, we will then consider that the realization $\{x^*(t,\omega),y^*(t,\omega)\}$ does not exist for $t \geqslant t(\omega)$.

Now, if $\varphi(t,x,y)$ is a certain scalar function, each realization $\{x^*(t,\omega),y^*(t,\omega)\}$ generates a realization of the random function $\varphi(t)$ with the appropriate probability distribution, where we assume that

$$\varphi(t,\omega) = \begin{cases} \varphi(t, x, (t, \omega), y(t, \omega)) & \text{for } t_0 \leqslant t < t(\omega) \\ \varphi(t(\omega), x(t(\omega), \omega))\, y(t(\omega, \omega)) & \text{for } t \geqslant t(\omega) \end{cases} \tag{2.17}$$

Let $\varphi_t = M[\varphi(t) / x_0, y_0]$ be the mathematical expectation of the random function $\varphi(t)$. Then the equality *

$$\left(\frac{d\varphi_t}{dt}\right)_{dt=+0} = M\left[\frac{dM[\varphi]}{dt} \Big/ x_0, y_0\right] \tag{2.18}$$

is valid.

Let us introduce the notation

$$P\{x^*(t) \in E / x_0, y_0\} = p(t) \tag{2.19}$$

Then we will have by virtue of (2.14)

$$P\{x^*(t) \in D / x_0, y_0\} \geqslant {}^1/_4\gamma - p(t) \tag{2.20}$$

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* The equality (2.18) is proved rigorously on the basis of the theory of infinitesimal operators of Markov processes [13]. Certain considerations for the derivation of this formula will be given below.

Taking account of (2.15) and (2.18), we have

$$\frac{dv_t}{dt} = M\left[\frac{dM\ [v]}{dt} \Big/ x_0, y_0\right] \leqslant -k^2 p(t)$$

Integrating and taking into account that $v_t \geqslant 0$ for all $t \geqslant t_0$, we obtain

$$\int_{t_0}^{\infty} p(t)\, dt \leqslant \frac{v_0}{k^2} \qquad (v_0 = v(t_0, x_0, y_0)) \tag{2.21}$$

On the other hand, taking account of conditions (2.16) and (2.20) as well as the boundedness of the derivative $dM[F]/dt$ in the domain (2.2), we will have

$$\frac{dF_t}{dt} = M\left[\frac{dM\ [F]}{dt} \Big/ x_0, y_0\right] \geqslant m^2\left[\frac{1}{4}\gamma - p(t)\right] - Kp(t)$$

Integrating this inequality and taking account of (2.16), we obtain $\lim F_t = \infty$ as $t \to \infty$, which is impossible since F_t is a bounded function. The contradiction obtained proves the validity of (2.7) and, consequently, of Theorem 2.1 .

Let us set down the proof of equality (2.18).

Let L denote the open domain $\delta < \|x\| < H$. Let us take the arbitrary number $0 < \nu < \frac{1}{2}(H - \delta)$ and let us define a closed domain S by conditions

$$S = \{\delta + \nu \leqslant \|x\| \leqslant H - \nu\}$$

Let us calculate the increment $\Delta\varphi_t$ in the function

$$\varphi_t = M\ [\varphi(t, x^*(t), y^*(t)) \,/\, x_0, y_0].$$

To do this, let us consider the following incompatible events which generate a complete group.

The event A , the solution $\{x^*(t), y^*(t)\}$ is not terminated at any point of the time interval $(t_0, t]$, where $x^*(t) = x(t) \in S$.

The event B , the solution $\{x^*(t), y^*(t)\}$ is not terminated at any point of the time interval $(t_0, t]$ but $x^*(t) = x(t) \in L \setminus S$.

The event C , the solution $\{x^*(t), y^*(t)\}$ terminates at a certain time $\tau \in (t_0, t]$

Then we will have (for $\Delta t > 0$)

$$\Delta\varphi_t = \varphi_{t+\Delta t} - \varphi_t = M\ [\varphi(t + \Delta t) - \varphi(t) \,/\, x_0, y_0 A] + \\ + M\ [\varphi(t + \Delta t) - \varphi(t) \,/\, x_0, y_0, B] + M\ [\varphi(t + \Delta t) - \varphi(t) \,/\, x_0, y_0, C] \tag{2.22}$$

Let us estimate each of the terms. The right-hand sides of Equations (1.1) satisfy the Lipschitz conditions, hence with the compliance of condition A it is possible to indicate so small a value of Δt , that the solution $\{x^*(t), y^*(t)\}$ will not terminate even in the time interval $(t, t+\Delta t)$. Then applying the formula of repeated mathematical expectations ([5], p.40), we will have

$$M\ [\varphi(t + \Delta t) - \varphi(t) \,/\, x_0, y_0, A] = M\ [M\ [\varphi(t + \Delta t,\ x(t + \Delta t),\ y(t + \Delta t) - \\ - \varphi(t, x(t), y(t)) / x(t) = \xi \in S,\ \ y(t) = \\ = \eta \in Y \,/\, x_0, y_0, A]] = M\left[\frac{dM\ [\varphi]}{dt} \Big/ x_0, y_0, A\right]\Delta t + o(\Delta t) \tag{2.23}$$

The estimate (K is a certain constant)

$$|\, M\ [\varphi(t + \Delta t) - \varphi(t) \,/\, x_0, y_0, B]\,| \leqslant K\Delta t\, P\, \{x^*(t) \in L \setminus S \,/\, x_0, y_0\} + o(\Delta t) \tag{2.24}$$

is valid for th: second term on the right-hand side of (2.22).

Finally, it is clear from the definition of the function $\varphi(t)$ that

$$M\,[\varphi\,(t+\Delta t)-\varphi\,(t)\,/\,x_0,\,y_0,\,C]=0 \tag{2.25}$$

Hence, it follows from the estimates (2.23), (2.24) and (2.25) that the function $\varphi_t = M\,[\varphi\,(t)\,/\,x_0,\,y_0]$ is continuous on the right in t uniformly in the domain $t \geqslant t_0$ and, consequently, it is continuous in this domain. Now taking into account that

$$\lim P\,\{x^*\,(t)\in L\setminus S\,/\,x_0,\,y_0\}=0 \qquad \text{for } \nu\to 0$$

we obtain the equality (2.18).

3. As an example, let us consider the second order equation

$$x^{\cdot\cdot}+ax^{\cdot}+bx=0 \tag{3.1}$$

which is equivalent to the system

$$x^{\cdot}=z, \qquad z^{\cdot}=-bx-az \tag{3.2}$$

Here $a(y)$ and $b(y)$ are known bounded functions of the variable y and $y(t)$ describes a homogeneous Markov stochastic process with a finite number of states $Y\{y_1,\ldots,y_r\}$, the elements of the transition matrix $p_{ij}(\Delta t)$ are given within the time Δt by Formulas

$$p_{ij}\,(\Delta t)=\alpha_{ij}\,\Delta t+o\,(\Delta t) \qquad (i\neq j,\ \alpha_{ij}=\text{const},\ i,j=1,\ldots,r) \tag{3.3}$$

Here $p_{ij}(\Delta t)$ is the probability of the change in values $y_i \to y_j$ during the time Δt.

It is known that compliance with the inequalities $a>0$, $b>0$ will be the necessary and sufficient condition for stability of the system (3.2) with a = const and b = const in the determined case. Let us assume that

$$b\,(y)>0 \qquad \text{for } y\in Y \tag{3.4}$$

Let us introduce the notation

$$a(y_k)=a_k, \qquad b(y_k)=b_k>0 \qquad\qquad (k=1,\ldots,r)$$

Now, let us consider the positive definite function

$$v\,(x,\,z,\,k)=x^2+\frac{1}{b_k}z^2 \tag{3.5}$$

To evaluate $dM[v]/dt$ by virtue of Equations (3.2) at the point $\{x,z,k\}$ let us use the ecuality ([11], English version p.1229)

$$\frac{dM\,[v]}{dt}=\frac{\partial v}{\partial x}z+\frac{\partial v}{\partial z}(-b_kx-a_kz)+\sum_{j\neq k}^{r}\alpha_{kj}\,[v\,(x,\,k,\,j)-v\,(x,\,z,\,k)] \tag{3.6}$$

After some transformations we will have

$$\frac{dM\,[v]}{dt}=-\,z^2\left\{\frac{2a_k}{d_k}-\sum_{j\neq k}^{r}\left(\frac{1}{b_j}-\frac{1}{b_k}\right)\alpha_{kj}\right\} \tag{3.7}$$

Let us now require compliance with conditions

$$\frac{2a_k}{b_k}-\sum_{j\neq k}^{r}\left(\frac{1}{b_j}-\frac{1}{b_k}\right)\alpha_{kj}>0 \qquad\qquad (k=1,\ldots,r) \tag{3.8}$$

Then $dM[v]/dt$ will be negative, it may vanish on the line $z = 0$. Let us construct the function

$$F(x, z) = -xz \tag{3.9}$$

Then a domain G containing the line $z = 0$ may be indicated such that $dM[F]/dt$ will be positive definite in this domain G by virtue of Equations (3.2) since for $z = 0$

$$\frac{dM[F]}{dt} = b_k x^2 > 0$$

Thus for the asymptotic stability in probability in the large for the system (3.2), compliance with conditions (3.4) and (3.8) is sufficient. It is seen, in particular, from these conditions that the asymptotic stability in probability in the large may hold even when certain of the possible values of a_k will be negative or zero.

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